MATH 2040 Lecture 23 (1/12/2016)

§ Existence of Jordan canonical basis

<u>Recall:</u> Given $T : V \to V$, $\mathbb{F} = \mathbb{C}$, $\dim V < +\infty$.

$$V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k}$$
$$\beta = \beta_1 \cup \cdots\cdots \cup \beta_k$$

<u>Theorem:</u> Each generalized eigenspace K_λ has a basis consisting of union of "cycles" of the form

$$\gamma = \{ \underbrace{(T-\lambda I)^{p-1} v}_{\substack{\text{initial vector} \\ \in E_\lambda}}, \ldots\ldots, (T-\lambda I) v, \underbrace{v}_{\text{end vector}} \}$$

<u>Recall:</u>

$$A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \rightsquigarrow A - \lambda I = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

$$(A-\lambda I)^2 = \begin{pmatrix} 0 & \cancel{0} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix}$$

$$\gamma = \{(T-\lambda I)^{p-1}v, \ldots\ldots, (T-\lambda I)v, v\}$$

$$\downarrow\ T-\lambda I$$

$$0 \qquad \{(T-\lambda I)^{p-1}v, \ldots\ldots, (T-\lambda I)v\}$$

<u>Lemma</u>: Suppose we have a collection of cycles in K_λ:

$$\gamma_1 = \{(T-\lambda I)^{p_1-1}v_1, \ldots\ldots, (T-\lambda I)v_1, v_1\}$$

$$\vdots \qquad\qquad \vdots$$

$$\gamma_k = \{(T-\lambda I)^{p_k-1}v_k, \ldots\ldots, (T-\lambda I)v_k, v_k\}$$

forms a linearly indep set in E_λ

Then, $\gamma = \gamma_1 \cup \ldots \cup \gamma_k$ is linearly indep.!

<u>Proof</u>: Induction on $|\gamma| :=$ # elements in γ

- $|\gamma| = 1$: trivial
- Assume holds for $|\gamma| \leq n-1$.
- Let $|\gamma| = n$. (<u>Note</u>: γ lin. indep $\Leftrightarrow$ dim span $\gamma = n$)

Define $W = \text{span}\,\gamma$. <u>Note</u>: · $\dim W \leq n$

· W is $(T-\lambda I)$-invariant.

Restrict the operator $T : K_\lambda \to K_\lambda$

$$\mathcal{U} := T - \lambda I\big|_W : \begin{array}{ccc} W & \longrightarrow & W \\ \cup\!\!\shortmid & & \cup\!\!\shortmid \\ W \cap E_\lambda & \longrightarrow & 0 \end{array}$$

Observe: $R(\mathcal{U}) = \text{span}$

$$\gamma' = \left\{ \begin{array}{l} \gamma_1' = \{ (T-\lambda I)^{p_1 - 1} v_1, \ldots\ldots, (T-\lambda I) v_1 \} \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ \gamma_k' = \{ (T-\lambda I)^{p_k - 1} v_k, \ldots\ldots, (T-\lambda I) v_k \} \end{array} \right\}$$

lin. indep.

$\Rightarrow |\gamma'| = n - k < n$

$\xRightarrow[\text{hypothesis}]{\text{induction}}$ γ' lin. indep. $\Rightarrow$ $\boxed{\dim R(\mathcal{U}) = n - k.}$

So, $\boxed{\dim N(\mathcal{U}) \geq k}$

Rullity - Nullity Thm $\Rightarrow$

$$n \geq \dim W = \underbrace{\dim R(\mathcal{U})}_{= \, n-k} + \underbrace{\dim N(\mathcal{U})}_{\geq \, k} \geq n$$

$\Rightarrow \dim W = n.$

$\square$

Proof of Thm on $\exists$ of basis consisting of cycles:

Induction on $\dim K_\lambda = n$.

- $n=1$: $K_\lambda = E_\lambda$ clear!
- Assume holds for $\dim K_\lambda \leq n-1$.
- Now, suppose $\dim K_\lambda = n$.

$$U := T - \lambda I\big|_{K_\lambda} : K_\lambda \longrightarrow K_\lambda$$

Consider $W = R(U) \subseteq K_\lambda$ (U-inv.)

Since $N(U) = E_\lambda \neq 0 \Rightarrow R(U) \neq K_\lambda$

$$S := U\big|_W : W \longrightarrow W$$

Apply induction hypothesis $\Rightarrow$ $W = R(U)$ has a basis consisting of union of cycles

$$\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_q \quad \text{for } W$$

Write things out very explicitly:

$W = R(U) = R(T-\lambda I)$

γ

$\gamma_1' = \quad \gamma_1 = \{(T-\lambda I)^{p_1-1}v_1, \dots, v_1\} \cup \{v_1'\}$, $\quad v_1 \xleftarrow{U} v_1'$

$\gamma_2' = \quad \gamma_2 = \{(T-\lambda I)^{p_2-1}v_2, \dots, v_2\} \cup \{v_2'\}$, $\quad v_2 \xleftarrow{U} v_2'$

$\vdots \qquad \vdots \qquad \vdots$

$\gamma_q' = \quad \gamma_q = \{(T-\lambda I)^{p_k-1}v_k, \dots, v_k\} \cup \{v_k'\}$, $\quad v_k \xleftarrow{U} v_k'$

$\gamma_{q+1}' = \{w_1\}$

$\vdots$

$\gamma_{q+r}' = \{w_r\}$

basis (lin. indep.)

E_λ

Claim: $\gamma' = \bigcup_{j=1}^{q+r} \gamma_j'$ is a basis for K_λ

Pf: γ' is lin. indep. by Lemma.

Check: $|\gamma'| = \dim K_\lambda = n$

Pf: $|\gamma| = \dim W = \dim R(U) = m$

and $|\gamma'| = m + q + r \overset{?}{=} n$ ✓

Rank-nullity apply $U : K_\lambda \to K_\lambda$

$$n = \dim K_\lambda = \underbrace{\dim N(U)}_{\substack{\dim E_\lambda \\ \parallel \\ q+r}} + \underbrace{\dim R(U)}_{\substack{\dim W \\ \parallel \\ m}} = m+q+r$$

$\blacksquare$